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# Braid group representation associated with the 10-dimensional representation of SU(5) and its Yang–Baxterisation

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**Abstract.** A braid group representation and the corresponding $q$-exterior algebra associated with the 10-dimensional representation of SU(5) are calculated. The method involves consideration of weight conservation, Casimir eigenvalues and the property of Markov trace for Jimbo–Reshetikhin-type solutions. Applying a recently developed procedure to this spectral-independent solution to the Yang–Baxter equation, the trigonometric spectral-dependent solution is found.

## 1. Introduction

Remarkable progress has been made in the derivation of trigonometric solutions of the quantum Yang–Baxter equations (YBE) [1–5] associated with classical Lie algebras. The sl(2) case has been solved for arbitrary representations [6]; in addition, the spectral-*dependent* solutions associated with the *fundamental* representation of $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$ and $D_{2n+1}^{(2)}$ have been solved by Jimbo [7], and recently $G_2$ by Kuniba [8], on the basis of quantum groups (QG). However, this beautiful theory does not automatically extend to higher representations. Recently, Reshetikhin [9] gave a general method for formulating the spectral-*independent* solutions associated with an arbitrary representation of a classical Lie algebra; he showed that the braid group representation (BGR) for any simple Lie algebra can be calculated based on the corresponding Casimir eigenvalues. (This approach is closely related to CFT through Witten's discussion [10, 11].) This procedure, however, involves computing the $q$-analogue Clebsch–Gordan coefficient, which become cumbersome for non-fundamental representations.

In [12], a systematic approach was proposed for practical calculation of the *spectral-dependent* solutions of YBE associated with a higher representation of a simple Lie algebra. This approach consists of two essential steps: (i) calculate the solution of the BGR, based on the theorem proved in [9], but without calculating $q$-Clebsch–Gordan coefficients [13]; (ii) 'Yang–Baxterise' these solutions, i.e., perform a 'deformation' to the spectral dependent solutions; here a hint from Jimbo's work has been used [6, 12]. In the present work, this procedure is illustrated through an explicit calculation; we

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apply it to the case of the 10-dimensional representation of SU(5). An outline of the procedure is given in the following.

The objective is to solve the spectral-dependent YBE

$$(\check{\mathbf{R}}_{12}(x) \otimes \mathbf{I})(\mathbf{I} \otimes \check{\mathbf{R}}_{23}(xy))(\check{\mathbf{R}}_{12}(x) \otimes \mathbf{I}) = (\mathbf{I} \otimes \check{\mathbf{R}}_{23}(y))(\check{\mathbf{R}}_{12}(xy) \otimes \mathbf{I})(\mathbf{I} \otimes \check{\mathbf{R}}_{23}(x)) \tag{1.1}$$

where $x, y$ are related to the usual spectral parameters, $u$, by $x = e^{-u}$. In addition, the unitarity condition

$$\check{\mathbf{R}}(x)\check{\mathbf{R}}(x^{-1}) = \rho(x)\mathbf{I} \tag{1.2}$$

and the 'initial' condition

$$\check{\mathbf{R}}(1) \propto \mathbf{I} \tag{1.3}$$

are imposed. The unitarity condition is satisfied by all the solutions found by Jimbo [7]; and the initial condition corresponds to the physical requirement that the $R(x)$-matrix, which is related to $\check{\mathbf{R}}(x)$ by $\check{\mathbf{R}}(x) = P\mathbf{R}(x)$ and $P(X \otimes Y) = Y \otimes X$, is reduced to a permutation in this limit [14]. The corresponding $x$-independent solution, $\mathbf{S} = \check{\mathbf{R}}(0)$, as $u \to \infty$, satisfies

$$\sum_{\mu\nu\rho} S_{\alpha\beta}^{\mu\nu} S_{\nu\gamma}^{\rho\kappa} S_{\mu\rho}^{\lambda\omega} = \sum_{\mu\nu\rho} S_{\beta\gamma}^{\mu\nu} S_{\alpha\mu}^{\lambda\rho} S_{\rho\nu}^{\omega\kappa} . \tag{1.4}$$

This is the equation for a BGR.

The first part of the calculation is to solve equation (1.4) associated with a given representation of a simple Lie algebra $R$. The first task is to suitably label the matrix $\mathbf{S}$; this is subject to the extra conservation laws one wishes to impose. Here we consider the 'six-vertex-type' solutions, characterised by 'weight conservation'. This consideration fixes the structure of $\mathbf{S}$ and determines part of the matrix elements; in general $\mathbf{S}$ is decomposed into a block-diagonal matrix with a large number of vanishing elements. This step does not involve the YBE.

Reshetikhin [9] proved the following: given the standard decomposition into irreducible subspaces of $R \otimes R$

$$R \otimes R = \oplus_i^N E_i . \tag{1.5}$$

The corresponding BGR, $\mathbf{S}$, can be decomposed into

$$\mathbf{S} = \sum_{i=1}^{N} \lambda_i P_i \tag{1.6}$$

where $\lambda_i$ are the unequal eigenvalues of $\mathbf{S}$, and $P_i$ the corresponding projectors; the eigenvalues can immediately be computed by the classical Casimirs with the simple formula, in our notation

$$\lambda_i = \pm q^{2C_R - C_{E_i}} \tag{1.7}$$

where $C_R$ and $C_{E_i}$ are the Casimirs of the representation $R$ and $E_i$, respectively, and the + (−) sign is chosen if the representation is symmetric (antisymmetric).

The non-zero elements of $\mathbf{S}$ are then partially fixed by computing the determinant and the trace of each submatrices, using the known eigenvalues computed from (1.7) and the Markov trace property [9, 15]. Finally the rest of the undetermined elements are found by direct substitution in (1.4), via a diagrammatic expansion [16].

The above steps produce an $\mathbf{S}$ satisfying (1.4). Next we apply the Yang–Baxterisation scheme of [17] to generate the spectral-dependent solutions of YBE, namely $\check{\mathbf{R}}(x)$ with arbitrary $x$.

We remark that the above procedure has been shown to be successful not only for the Jimbo–Reshetikhin (JR)-type solutions, but also for non-JR-type solutions; the former includes higher-dimensional representations such as the eight-dimensional representation of $B_3$ and the six-dimensional representation of SU(3) [13], and the latter includes new solutions for $B_2, C_2$ and $D_2$ [18].

The rest of this paper is organised as follows. In section 2, we perform some standard group-theoretic calculation for the $10\otimes 10$ representation of SU(5) to determine the general structure for $\mathbf{S}$ constrained by the condition of weight conservation; here we assign a 'pseudospin' labelling to the 10-dimensional representation of SU(5). Next, in section 3, we determine $\mathbf{S}$ which satisfies the spectral-independent YBE, or equation (1.4); Reshetikhin's theorems are exploited to simplify the calculation. The results are contained in (2.4), (2.5), (3.8) and (3.9). The eigenvectors of $\mathbf{S}$ are computed in section 4, and are given in (4.1) and (4.2); we also check the classical limit and find complete agreement with general theory of QG [9]. The $q$-analogue algebraic structure generated by $\mathbf{S}$ are given in section 5. Here the usual $q$-analogue algebra found for fundamental representations are reproduced, but some new commutation relations peculiar to the higher representation are also found, as shown in (5.8)–(5.10). In the last section, we perform the Yang–Baxterisation scheme to find $\check{\mathbf{R}}(x)$, the solution to the spectral-dependent YBE; the results are contained in (6.11)–(6.15). Some illustrations for section 6 are given in the appendix. A note on the notation used: with the exception of (6.14), blank spaces in all matrices represent zero elements.

## 2. Weight conservation

The 10-dimensional representation of SU(5) is characterised by the 10 weights $\boldsymbol{w}_l$; they are given as follows in the Cartan–Weyl basis of the five-dimensional root space:

$$\begin{aligned}
&\boldsymbol{w}_1 = \tfrac{1}{5}(3,3,-2,-2,-2) && \boldsymbol{w}_2 = \tfrac{1}{5}(3,-2,3,-2,-2)\\
&\boldsymbol{w}_3 = \tfrac{1}{5}(3,-2,-2,3,-2) && \boldsymbol{w}_4 = \tfrac{1}{5}(-2,3,3,-2,-2)\\
&\boldsymbol{w}_5 = \tfrac{1}{5}(3,-2,-2,-2,3) && \boldsymbol{w}_6 = \tfrac{1}{5}(-2,3,-2,3,-2)\\
&\boldsymbol{w}_7 = \tfrac{1}{5}(-2,-2,3,3,-2) && \boldsymbol{w}_8 = \tfrac{1}{5}(-2,3,-2,-2,3)\\
&\boldsymbol{w}_9 = \tfrac{1}{5}(-2,-2,3,-2,3) && \boldsymbol{w}_{10} = \tfrac{1}{5}(-2,-2,-2,3,3)\,.
\end{aligned} \tag{2.1}$$

The BGR $\mathbf{S}$ is a $100\times 100$ matrix acting on the $10\otimes 10$ tensor space. Since we impose the condition of weight conservation, it is convenient to label the states in the 10-dimensional representation with a 'pseudospin', in analogy with the quantum number $s_z$ for the well known SU(2) case. This labelling is designed to reflect the following conditions:

$$\begin{aligned}
&\boldsymbol{w}_1 + \boldsymbol{w}_7 = \boldsymbol{w}_2 + \boldsymbol{w}_6 = \boldsymbol{w}_3 + \boldsymbol{w}_4\\
&\boldsymbol{w}_1 + \boldsymbol{w}_{10} = \boldsymbol{w}_3 + \boldsymbol{w}_8 = \boldsymbol{w}_5 + \boldsymbol{w}_6
\end{aligned}$$

$$w_3 + w_9 = w_2 + w_{10} = w_5 + w_7 \tag{2.2}$$
$$w_1 + w_9 = w_4 + w_5 = w_2 + w_8$$
$$w_4 + w_{10} = w_6 + w_9 = w_7 + w_8 .$$

These are all the conserved pairs. Up to a factor, the above determines the pseudospin asignment; we set

| State | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ | $w_9$ | $w_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Pseudospin | $\frac{11}{2}$ | $\frac{9}{2}$ | $\frac{5}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{3}{2}$ | $-\frac{5}{2}$ | $-\frac{7}{2}$ | $-\frac{11}{2}$ . |

(2.3)

It is easy to check that they satisfy (2.2). Note that, unlike in the case with fundamental representation, these pseudospins are not uniformly distributed – $\frac{7}{2}$ and $-\frac{9}{2}$ do not exist. Thus the requirement that the total pseudospins be conserved forces the matrix $\mathbf{S}$ into block-diagonal form

$$\mathbf{S} = \sum_{\substack{k=-11 \\ k\neq -10}}^{11} \oplus \mathbf{A}_k \tag{2.4}$$

where $\mathbf{A}_k$ are the submatrices acting on the subspace in which the total pseudospin is $k$. For example $\mathbf{A}_{-7}$ acts on the three-dimensional space consisting of $(-\frac{3}{2}, -\frac{11}{2})$, $(-\frac{7}{2}, -\frac{7}{2})$ and $(-\frac{11}{2}, -\frac{3}{2})$.

By standard method in group representation theory, these submatrices are found to have three different eigenvalues, $\lambda_1, \lambda_2, \lambda_3$ with multiplicity $n_1, n_2$ and $n_3$, respectively; denoting this as $(n_1, n_2, n_3)$ we have

$$\begin{array}{ll}
\mathbf{A}_{\pm 11}, \mathbf{A}_9 : (1,0,0) & \mathbf{A}_{-2} : (3,3,0) \\
\mathbf{A}_{10}, \mathbf{A}_{-9}, \mathbf{A}_{\pm 8} : (1,1,0) & \mathbf{A}_{\pm 3}, \mathbf{A}_1 : (4,3,0) \\
\mathbf{A}_{-7} : (2,1,0) & \mathbf{A}_{\pm 4} : (2,3,1) \\
\mathbf{A}_7, \mathbf{A}_{\pm 6} : (2,2,0) & \mathbf{A}_2, \mathbf{A}_0 : (3,4,1) \\
\mathbf{A}_{\pm 5} : (3,2,0) & \mathbf{A}_{-1} : (4,4,1) .
\end{array} \tag{2.5}$$

It is easy to check that the total multiplicities of $\lambda_1, \lambda_2, \lambda_3$ are 50, 45 and 5, respectively. This is expected from the decomposition into irreducible subspaces of SU(5)

$$10 \otimes 10 = 50_S + 45_A + 5_S . \tag{2.6}$$

Thus the three eigenvalues correspond to the 50-dimensional symmetric, 45-dimensional antisymmetric and five-dimensional symmetric subspces, respectively.

The weight conservation (2.2) is more restrictive than their pseudospin assignment suggests; for example, $(\frac{3}{2}, -\frac{3}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$ both have pseudospin 0, but the corresponding weights are *not* conserved, $w_4 + w_7 \neq w_5 + w_6$. Therefore, many elements in each submatrix $\mathbf{A}_k$ vanishes by weight conservation. For those $\mathbf{A}_k$ with only two unequal eigenvalues, all the non-zero elements lie along the skew diagonal and the lower half of the diagonal; for example

$$\mathbf{A}_5 = \begin{array}{c} \\ (\frac{11}{2},-\frac{1}{2}) \\ (\frac{9}{2},\frac{1}{2}) \\ (\frac{5}{2},\frac{5}{2}) \\ (\frac{1}{2},\frac{9}{2}) \\ (-\frac{1}{2},\frac{11}{2}) \end{array} \begin{array}{c} \begin{array}{ccccc} (\frac{11}{2},-\frac{1}{2}) & (\frac{9}{2},\frac{1}{2}) & (\frac{5}{2},\frac{5}{2}) & (\frac{1}{2},\frac{9}{2}) & (-\frac{1}{2},\frac{11}{2}) \end{array} \\ \begin{pmatrix} & & & & \ast \\ & & & \bullet & \\ & & \ast & & \\ & \ast & & \ast & \\ \ast & & & & \ast \end{pmatrix} \end{array} . \tag{2.7}$$

Note that we follow the convention of ordering the state according to the magnitude of the first pseudospin. The submatrices $\mathbf{A}_{\pm 4}, \mathbf{A}_2, \mathbf{A}_0$ and $\mathbf{A}_{-1}$ contain the third eigenvalue, and do not have the above form; the upper-left triangle of these matrices still vanishes as in (2.7), but the lower-right triangle do not vanish in general. In addition, $\mathbf{A}_2, \mathbf{A}_0$ and $\mathbf{A}_{-1}$ are essentially direct sums: the largest submatrix $\mathbf{A}_{-1}$ has the form

$$\mathbf{A}_{-1} = \begin{pmatrix} & & & & & & & & \ast \\ & & & & & & & \ast & \ast \\ & & & & & & \diamond & & \\ & & & & & \ast & & \ast & \ast \\ & & & & \bullet & & & & \\ & & & \ast & & \ast & & \ast & \ast \\ & & \diamond & & & & \diamond & & \\ & \ast & & \ast & & \ast & & \ast & \ast \\ \ast & \ast & & \ast & & \ast & & \ast & \ast \end{pmatrix} \tag{2.8}$$

and the eight-dimensional $\mathbf{A}_2, \mathbf{A}_0$ are obtained by removing the centre row and column from (2.8). We shall refer to the first category as the fundamental-type submatrices, since this is characteristic of the fundamental representations of SU($n$).

## 3. The braid group representation

The $100 \times 100$ matrix $\mathbf{S}$ has been greatly simplified by the weight conservation consideration of section 2; so far, the YBE has not been involved. Next we find conditions to reduce the number of independent elements of $\mathbf{S}$.

The Casimirs of each subspace in the decomposition (1.5) can be easily calculated by standard method; then by equations (1.5)–(1.7), we immediately find that

$$\lambda_1 = q^{-6/5} \qquad \lambda_2 = -q^{4/5} \qquad \lambda_3 = q^{24/5} . \tag{3.1}$$

For simplicity, we renormalise $\mathbf{S}$ so that the eigenvalues become

$$\lambda_1 = 1 \qquad \lambda_2 = -t^2 \qquad \lambda_3 = t^6 \tag{3.2}$$

where $t$ is an arbitrary ($q$-analogue) parameter in place of $q$. Since the eigenvalues of each submatrix of $\mathbf{S}$ are known from (2.5), the trace and the determinant impose two conditions on the non-zero elements of each $\mathbf{A}_k$ in terms of the parameter $t$.

Next we use the requirements of the Markov trace to impose further conditions. It is known [9] that for JR-type solutions of the BGR, following property holds: there exists an $h$-matrix given by

$$h_{ab} = \delta_{ab}\, t^{-2\langle w_a \cdot \boldsymbol{\eta}\rangle} \tag{3.3}$$

where $\boldsymbol{\eta}$ is the half-sum of positive roots, such that

$$\sum_b S_{ab}^{ab} h_{bb} = f(t) \qquad a = -\tfrac{11}{2}, -\tfrac{7}{2}, -\tfrac{5}{2}, \ldots \tag{3.4}$$

where $f(t)$ is independent of the index $a$. In the present case

$$\boldsymbol{\eta} = (2, 1, 0, -1, -2) \tag{3.5}$$

and hence

$$\mathbf{h} = \mathrm{diag}(t^{-6}, t^{-4}, t^{-2}, t^{-2}, 1, 1, t^2, t^2, t^4, t^6)\,. \tag{3.6}$$

Thus we obtain another nine conditions for $\mathbf{S}$. Note that the terms in (3.4) come from different submatrices.

The remaining undetermined elements are calculated by explicit use of the YBE; we use an extended diagrammatic approach [16] which involves calculating a few diagrams only. By this procedure we find the submatrices of $\mathbf{S}$ as follows: let

$$w \equiv 1 - t^2 \qquad z \equiv 1 + t^2 \tag{3.7}$$

the fundamental-type submatrices have the form

$$\mathbf{A}_5 = \begin{pmatrix} & & & & t \\ & & & t & \\ & & 1 & & \\ & t & & w & \\ t & & & & w \end{pmatrix} \tag{3.8}$$

that is, all the non-zero diagonal elements are equal to $w$ and the skew-diagonal elements equal $t$, except that the centre element is equal to 1 for odd dimensional matrices. The non-fundamental-type submatrix $\mathbf{A}_{-1}$ is given by

$$\mathbf{A}_{-1} = \begin{pmatrix} & & & & & & & & t^2 \\ & & & & & & & t^2 & tw \\ & & & & & & t & & \\ & & & & & t^2 & & tw & -t^2w \\ & & & & 1 & & & & \\ & & & t^2 & & 0 & & tw & -t^2w \\ & & t & & & & w & & \\ & t^2 & & tw & & tw & & w^2 & t^3w \\ t^2 & tw & & -t^2w & & -t^2w & & t^3w & zw^2 \end{pmatrix}. \tag{3.9}$$

The eight-dimensional $\mathbf{A}_0$ and $\mathbf{A}_2$ are obtained by removing the centre row and column; and the six-dimensional $\mathbf{A}_{\pm 4}$ are given by removing from (3.9) the lower-dimensional submatrices, i.e., by removing the third, fifth and seventh rows and columns.

We should emphasise that the above solution is only a JR-type solution. In principle, there may exist other solutions which are not of JR type. Next we check the classical limit and discuss the $q$-exterior algebraic structure of this solution [19].

## 4. Classical limit

The eigenvectors of $\mathbf{S}$ are calculated by computing those for the submatrices $\mathbf{A}_k$. For the fundamental-type submatrices, the eigenvectors are simple; arranging them in columns in the order that the $\lambda_1$-eigenvectors precede the $\lambda_2$-eigenvectors, we have, for example

$$\mathbf{L}_5 = \begin{array}{c} (\frac{11}{2},-\frac{1}{2}) \\ (\frac{9}{2},\frac{1}{2}) \\ (\frac{5}{2},\frac{5}{2}) \\ (\frac{1}{2},\frac{9}{2}) \\ (-\frac{1}{2},\frac{11}{2}) \end{array} \begin{pmatrix} t & & & & 1 \\ & t & & 1 & \\ & & 1 & & \\ & 1 & & -t & \\ 1 & & & & -t \end{pmatrix} . \tag{4.1}$$

Note that the ordering of the columns is arbitrary, depending on how one orders the eigenvalues; here we adopt the ordering $\lambda_1, \ldots, \lambda_2, \ldots, \lambda_3$. The even-dimensional submatrices have the same form, but do not contain the singlet state in the centre. It is easy to see that as $t \to 1$, every eigenvector is either symmetric (for $\lambda_1$) or antisymmetric (for $\lambda_2$), as expected, and is consistent with the multiplicity of $\lambda_1$ and $\lambda_2$, as shown in (2.4).

For the non-fundamental submatrices, we find for the six-dimensional submatrices

$$\mathbf{L}_{\pm 4} = \begin{pmatrix} t^2 z & 0 & 0 & t & -2t^2 & 1 \\ t & t^2 & 0 & 1 & 2t^3 & -t \\ -t^2 & t & 1 & 0 & wz & t^2 \\ -t^2 & t & -1 & 0 & wz & t^2 \\ t^3 & 1 & 0 & -t^2 & -2t & -t^3 \\ z & 0 & 0 & -t & 2t^2 & t^4 \end{pmatrix} . \tag{4.2}$$

The eight(nine)-dimensional submatrices are simply a direct sum of this and a two-dimensional (and a trivial) fundamental-type submatrix.

In the classical limit as $t \to 1$, the general theory for the JR-type solutions of YBE [9] requires that

$$\mathbf{S} \approx P(1 + (t-1)\mathbf{r}) \tag{4.3}$$

where $P$ is the permutation operator and $\mathbf{r}$ the Casimir operator. As a check, we can verify (4.3) for each submatrix. For example, we expand $\mathbf{A}_4$ (the 6×6 submatrix in equation (3.9)) to the first order in $t-1$ and factor out the permutation matrix $\mathbf{P}_4$, which is simply a skew-identity matrix, we find

$$\mathbf{A}_4 \approx \mathbf{P}_4(1 + (t-1)\mathbf{r}_4) \qquad \mathbf{r}_4 = -2 \begin{pmatrix} -1 & 1 & -1 & -1 & 1 & 0 \\ & -1 & 1 & 1 & 0 & 1 \\ & & -1 & 0 & 1 & -1 \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \\ & & & & & -1 \end{pmatrix} . \tag{4.4}$$

On the other hand, by setting $t = 1$ in $\mathbf{L}_4$, the normalised classical eigenvectors are simply

$$\phi_{1,1} = \frac{1}{\sqrt{12}}(2, 1, -1, -1, 1, 2)$$

$$
\begin{aligned}
\phi_{1,2} &= \frac{1}{\sqrt{4}}(0,1,1,1,1,0) \\
\phi_{2,1} &= \frac{1}{\sqrt{2}}(0,0,1,-1,0,0) \\
\phi_{2,2} &= \frac{1}{\sqrt{4}}(1,1,0,0,-1,-1) \\
\phi_{2,3} &= \frac{1}{\sqrt{4}}(-1,1,0,0,-1,1) \\
\phi_{3,1} &= \frac{1}{\sqrt{6}}(1,-1,1,1,-1,1)
\end{aligned}
\tag{4.5}
$$

where $\phi_{j,k}$ is the $k$th eigenvector associated with $\lambda_j$. Note that the parity of $\phi_{j,k}$ is even for $j = 1, 3$ and odd for $j = 2$. It is easy to check that for any $k$

$$
\phi^t_{j,k}\mathbf{r}_4\phi_{j,k} = \begin{cases} 0 & \text{for } j = 1 \\ 2 & \text{for } j = 2 \\ 6 & \text{for } j = 3 \end{cases}
\tag{4.6}
$$

in complete accord with the exponents appearing in (3.2).

## 5. The $q$-exterior algebraic structure

The BGR defines a $q$-exterior algebra through its decomposition into projection operators, i.e.

$$
\mathbf{S} = \sum_{i=1}^{3} \lambda_i P_i
\tag{5.1}
$$

where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are given in (3.2) and $P_i$ the projection operator given, in general, by

$$
P_j = \prod_{i \neq j}^{3} \left( \frac{\mathbf{S} - \lambda_i}{\lambda_j - \lambda_i} \right)
\tag{5.2}
$$

which projects out the eigenspace associated with $\lambda_i$ in the $10 \otimes 10$ dimensional representation. In the sector corresponding to $\lambda_i$, the $q$-exterior algebra is obtained by [19]

$$
\sum_{\gamma,\delta} (P_i)^{\gamma\delta}_{\alpha\beta} X_\gamma X_\delta = 0
\tag{5.3}
$$

where the indices run through the 10-pseudospin assignment in section 2. Since we have computed all the eigenvectors of $\mathbf{S}$, the projectors $P_i$ can be expressed easily. Let $\psi_{i,k,j}$ denote the normalised column eigenvector associated with $\lambda_i$ in the submatrix $\mathbf{A}_k$, and $j$ runs through the multiplicity in that submatrix and the row vector

$\mathbf{v}_k = (X_\alpha X_\beta, X_\gamma X_\delta, \ldots)$ corresponds to pseudospin assignment $((\alpha,\beta),(\gamma,\delta),\ldots)$ in $\mathbf{A}_k$. It is well known that

$$P_i = \sum_{k,j} \psi^t_{i,k,j} \psi_{i,k,j} \,. \tag{5.4}$$

Also, because the $\psi_{i,j,k}$ are orthogonal, equation (5.3) is equivalent to

$$\begin{aligned} &\psi_{i,k,j} \mathbf{v}^t_k = 0 && i = 1,2,3 \\ &k = 11, 10, \ldots, -9, -11 && \text{all } j\,. \end{aligned} \tag{5.5}$$

For the majority of the submatrices, the eigenvectors are decoupled, and hence we may separate the state vector $\mathbf{v}_k$ accordingly; for example, referring to (4.1), we have $\mathbf{v}_5 = (X_{11/2}X_{-1/2}, X_{-1/2}X_{11/2}) + (X_{9/2}X_{1/2}, X_{1/2}X_{9/2}) + (X_{5/2}X_{5/2})$. Clearly, all the eigenvectors related for the fundamental-type submatrices give rise to simple $q$-commutaion relations. This also applies to the low-dimensional sub-block of the non-fundamental-type $\mathbf{A}_k$; only the six-dimensional sub-block of $\mathbf{A}_{\pm 4}, \mathbf{A}_2, \mathbf{A}_0$ and $\mathbf{A}_{-1}$ may contain more complicated algebra. They are computed from equation (4.2) according to (5.5). The states affected are listed below (it is understood that the permuted states are included):

$$\begin{array}{llll} & (a,b) & (c,d) & (e,f) \\ \mathbf{A}_4 : & (\frac{11}{2}, -\frac{3}{2}) & (\frac{9}{2}, -\frac{1}{2}) & (\frac{5}{2}, \frac{3}{2}) \\ \mathbf{A}_{-4} : & (\frac{3}{2}, -\frac{11}{2}) & (-\frac{1}{2}, -\frac{7}{2}) & (-\frac{3}{2}, -\frac{5}{2}) \\ \mathbf{A}_2 : & (\frac{11}{2}, -\frac{7}{2}) & (\frac{9}{2}, -\frac{5}{2}) & (\frac{3}{2}, \frac{1}{2}) \\ \mathbf{A}_0 : & (\frac{11}{2}, -\frac{11}{2}) & (\frac{5}{2}, -\frac{5}{2}) & (\frac{1}{2}, -\frac{1}{2}) \\ \mathbf{A}_{-1} : & (\frac{9}{2}, -\frac{11}{2}) & (\frac{5}{2}, -\frac{7}{2}) & (\frac{1}{2}, -\frac{3}{2})\,. \end{array} \tag{5.6}$$

To summarise the results, define the $q$-exterior commutator and anticommutator

$$\{X, Y\}_t \equiv tXY + YX \qquad [X, Y]_t \equiv XY - tYX\,. \tag{5.7}$$

In the $\lambda_1$ sector, $X^2_\alpha = 0$ for all $\alpha$, and $\{X_\alpha, X_\beta\}_t = 0$ for $\alpha > \beta$, except if $(\alpha, \beta)$ is one of the case listed in (5.6); for those cases, they satisfy

$$\begin{aligned} &t\{X_c, X_d\}_1 + \{X_a, X_b\}_{t^2} = 0 \\ &\{X_c, X_d\}_{t^2} + t\{X_e, X_f\}_1 = 0\,. \end{aligned} \tag{5.8}$$

In the $\lambda_2$ sector, $[X_\alpha, X_\beta]_t = 0$ for $\alpha > \beta$, unless $(\alpha, \beta)$ is one of the cases listed in (5.6); for those cases, they satisfy

$$\begin{aligned} &[X_e, X_f] = 0 \\ &[X_c, X_d]_{t^2} + t[X_a, X_b]_1 = 0 \\ &wX_eX_f - t[X_c, X_d]_1 = 0\,. \end{aligned} \tag{5.9}$$

Finally, the $\lambda_3$ sector consists only of those states listed in (5.6). They satisfy

$$\{X_b, X_a\}_{t^4} - t[X_c, X_d]_{t^2} + t^2\{X_e, X_f\}_1 = 0\,. \tag{5.10}$$

## 6. Yang–Baxterisation

We have obtained the $x$-independent solution, $\mathbf{S}$, of the YBE for the 10-dimensional representation of SU(5). Next we apply the result of [12] to produce the corresponding $x$-dependent solution, $\check{\mathbf{R}}(x)$, for the BGR given in section 3. A brief summary of this Yang–Baxterisation procedure is given below.

For a BGR $\mathbf{S}$ associated with the fundamental representation, Jimbo [7] has shown that the corresponding $\check{\mathbf{R}}(x)$ has the form

$$\check{\mathbf{R}}(x) = \sum_{i=1}^{N} \rho_i(x) P_i \tag{6.1}$$

where $N$ is the number of unequal eigenvalues of $\mathbf{S}$ and $P_i$ are the corresponding projectors, and $\rho_i(x)$ is a polynomial in $x$ of degree $N-1$. In [12] it is shown that assuming the form (6.1) for the case of three unequal eigenvalues, only three possibilities are allowed which satisfy the unitarity condition, (1.2), and the initial condition, (1.3). They are

Case (a):
$$\begin{aligned} \rho_1(x) &= (x+\lambda_1\lambda_2^{-1})(x+\lambda_2\lambda_3^{-1}) \\ \rho_2(x) &= (\lambda_1\lambda_2^{-1}x+1)(x+\lambda_2\lambda_3^{-1}) \\ \rho_3(x) &= (\lambda_1\lambda_2^{-1}x+1)(\lambda_2\lambda_3^{-1}x+1) \end{aligned} \tag{6.2}$$

Case (b):
$$\begin{aligned} \rho_1(x) &= (x+\lambda_1\lambda_2^{-1})(x+\lambda_1\lambda_3^{-1}) \\ \rho_2(x) &= (\lambda_1\lambda_2^{-1}x+1)(x+\lambda_1\lambda_3^{-1}) \\ \rho_3(x) &= (x+\lambda_1\lambda_2^{-1})(\lambda_1\lambda_3^{-1}x+1) \end{aligned} \tag{6.3}$$

Case (c):
$$\begin{aligned} \rho_1(x) &= (x+\lambda_1\lambda_3^{-1})(x+\lambda_3\lambda_2^{-1}) \\ \rho_2(x) &= (\lambda_1\lambda_3^{-1}x+1)(1+\lambda_3\lambda_2^{-1}x) \\ \rho_3(x) &= (\lambda_1\lambda_3^{-1}x+1)(x+\lambda_3\lambda_2^{-1}). \end{aligned} \tag{6.4}$$

If the three cases are substituted in the $x$-dependent YBE, the $\mathbf{S}$ must satisfy the following equation:

$$f_3\Theta_3 + f_3'\Theta_3' + f_2\Theta_2 + f_1\Theta_1 + f_1'\Theta_1' = 0 \tag{6.5}$$

where the $\Theta$ are functions of the $\mathbf{S}$:

$$\begin{aligned} \Theta_3 &= S_i S_{i+1}^{-1} S_i - S_{i+1} S_i^{-1} S_{i+1} \\ \Theta_3' &= S_i^{-1} S_{i+1} S_i^{-1} - S_{i+1}^{-1} S_i S_{i+1}^{-1} \\ \Theta_2 &= S_i S_{i+1}^{-1} - S_{i+1} S_i^{-1} + S_{i+1}^{-1} S_i - S_i^{-1} S_{i+1} \\ \Theta_1 &= S_i - S_{i+1} \\ \Theta_1' &= S_i^{-1} - S_{i+1}^{-1} \end{aligned} \tag{6.6}$$

where $S_i, S_{i+1}$ are in the usual notation for braid groups such that the equation $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$ is equivalent to equation (1.4). The $f$ in (6.5) are functions of the eigenvalues which are given according to the cases as

Case (a):
$$f_3 = \frac{\lambda_1}{\lambda_3^2} \qquad f_3' = -\frac{\lambda_1^2}{\lambda_3}$$
$$f_2 = -\frac{\lambda_1}{\lambda_3}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_3} + \frac{\lambda_1}{\lambda_3}\right) \tag{6.7}$$
$$f_1 = -\lambda_2^{-1} f_2 \qquad f_1' = \lambda_2 f_2$$

Case (b):
$$f_3 = \frac{\lambda_1^3}{\lambda_2^2\lambda_3^2} \qquad f_3' = -\frac{\lambda_1^3}{\lambda_2\lambda_3}$$
$$f_2 = -\frac{\lambda_1^2}{\lambda_2\lambda_3}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_1^2}{\lambda_2\lambda_3}\right) \tag{6.8}$$
$$f_1 = -\lambda_1^{-1} f_2 \qquad f_1' = \lambda_1 f_2$$

Case (c):
$$f_3 = \frac{\lambda_1}{\lambda_2^2} \qquad f_3' = -\frac{\lambda_1^2}{\lambda_2}$$
$$f_2 = -\frac{\lambda_1}{\lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_3}{\lambda_2} + \frac{\lambda_1}{\lambda_3}\right) \tag{6.9}$$
$$f_1 = -\lambda_3^{-1} f_2 \qquad f_1' = \lambda_3 f_2 .$$

Since this procedure is independent of representations, we can use it to Yang–Baxterise our BGR.

We substitute our $S$ matrix in (6.5)–(6.9) for the three cases and find that case (a) is satisfied. A diagrammatic method has been used to perform this rather tedious but straightforward calculation; some illustration is given in the appendix. A similar calculation can also be found in [12], where the six-dimensional representation of SU(3) was computed.

Substituting (6.2) in (6.1) and using (5.2) we find an $x$-dependent solution to YBE for the 10-dimensional representation of SU(5).

$$\check{\mathbf{R}}(x) = \lambda_1 x(x-1)\mathbf{S}^{-1} + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_3}\right) x\mathbf{I} - \frac{1}{\lambda_3}(x-1)\mathbf{S} . \tag{6.10}$$

The inverse $\mathbf{S}^{-1}$ has the same block structure as $\mathbf{S}$, and can be obtained from $\mathbf{S}$ by 'reflecting' each submatrix along the skew-diagonal and letting $t \to t^{-1}$. The matrix $\check{\mathbf{R}}(x)$ is obviously of the block-diagonal form

$$\check{\mathbf{R}}(x) = \sum_{\substack{k=-11 \\ k\neq -10}}^{11} \oplus\, \check{\mathbf{R}}_k(x) \tag{6.11}$$

where each submatrix $\check{\mathbf{R}}_i(x)$ corresponds to the submatrix $\mathbf{A}_i$; but the $\check{\mathbf{R}}_i(x)$ no longer have the lower-triangular form as $\mathbf{A}_i$, although they remain symmetric. Corresponding to the fundamental-type $\mathbf{A}_i$, $\check{\mathbf{R}}_i(x)$ have the simple form, which we illustrate with $\check{\mathbf{R}}_5(x)$

$$\check{\mathbf{R}}_5(x) = \begin{pmatrix} xw_2 & & & & p_2 \\ & xw_2 & & p_2 & \\ & & w_0 & & \\ & p_2 & & w_2 & \\ p_2 & & & & w_2 \end{pmatrix} \tag{6.12}$$

where

$$\begin{aligned} p_2 &= t^{-1}(x - t^{-4})(x - 1) \\ w_0 &= (x - t^{-4})(x - t^{-2}) \\ w_2 &= (x - t^{-4})(1 - t^{-2}) \,. \end{aligned} \tag{6.13}$$

All the fundamental-type $\check{\mathbf{R}}_i(x)$ are of the above form, except that the even-dimensional ones do not contain the singlet $w_0$. The $\check{\mathbf{R}}_i(x)$ corresponding to the non-fundamental-type $\mathbf{A}_i$ include $\check{\mathbf{R}}_{\pm 4}(x)$,$\check{\mathbf{R}}_0(x)$, $\check{\mathbf{R}}_2(x)$ and $\check{\mathbf{R}}_{-1}(x)$. The largest submatrix is the 9×9 $\check{\mathbf{R}}_1(x)$; for simplicity we present only its lower-left triangle. ( Note that all $\check{\mathbf{R}}_i(x)$ are symmetric.)

$$\check{\mathbf{R}}_{-1}(x) = \begin{pmatrix} xw_1 & & & & & & & & \\ xu & xw_3 & & & & & & & \\ & & xw_2 & & & & & & \\ -xtu & xt^2u & & w_1 & & & & & \\ & & & & w_0 & & & & \\ -xtu & xt^2u & & p_1 & & w_1 & & & \\ & & p_2 & & & & w_2 & & \\ xt^2u & p_1 & & u & & u & & w_3 & \\ p_1 & u & & -tu & & -tu & & t^2u & x^{-1}w_1 \end{pmatrix} \tag{6.14}$$

where

$$\begin{aligned} p_1 &= t^{-2}(x - t^{-2})(x - 1) \\ w_1 &= x(1 - t^{-2})^2(1 + t^{-2}) \\ w_3 &= (1 - t^{-2})^2(x + t^{-2}) \\ u &= t^{-3}(1 - t^{-2})(x - 1) \,. \end{aligned} \tag{6.15}$$

As for the BGR, the eight-dimensional $\check{\mathbf{R}}_2(x)$ and $\check{\mathbf{R}}_0(x)$ are given by the above, except the centre row and column is removed; the six-dimensional $\check{\mathbf{R}}_{\pm 4}$ can be obtained by further removing the two-dimensional fundamental-type submatrix.

Thus we have found the $x$-dependent solution of YBE related to the 10-dimensional representation of SU(5). Using

$$R(x)_{ij,kl} = \check{R}(x)_{ij,lk} \tag{6.16}$$

the Boltzmann weights for the corresponding vertex models are obtained. Since there are a total of 310 non-zero elements in $R(x)$, this is a 310-vertex model.

We have shown by this particular example the process of generating the trigonometric solution of the YBE for a given representation of a simple Lie algebra which has one-multiplicity decomposition. Our approach can be satisfactorily applied to non-fundamental representations, and seems to be more convenient than the method by fusion rule [21–23], because the $q$-analogue projectors are computed without using the corresponding $q$-analogue Clebsch–Gordan (CG) coefficient matrices. In addition, our approach has the advantage of being applicable to the computation of new solutions to be presented elsewhere; these new solutions cannot be obtained by the usual fusion rule. As an illustration consider the two-dimensional representation of SU(2); the standard solution is related to the CG decomposition $2 \otimes 2 = 3 \oplus 1$, whereas the new solution corresponds to the decomposition $2 \otimes 2 = 2 \oplus 2$, i.e., a free fermion model. (See [24] for discussions for the cases $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$.)

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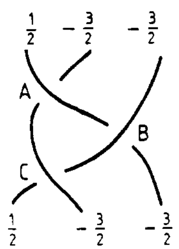
**Appendix**

The verification of (6.5) for case (a) is done by a diagrammatical expansion. We give a brief description of the method here.

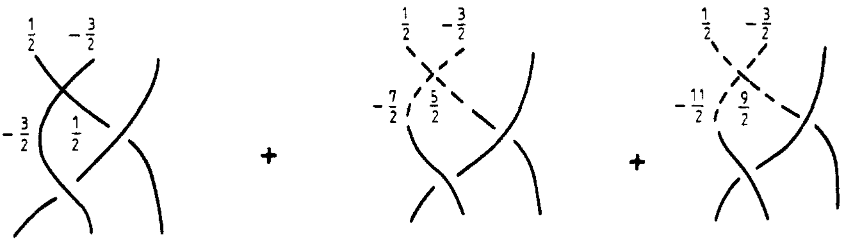
In terms of components, the equation (6.5) and (6.6) contain six free indices, as shown in (1.4), which represent three in-state indices, $\alpha, \beta, \gamma$ and three out-state indices, $\kappa, \lambda, \omega$. Because of the conservation condition, only the combinations satisfying

$$\alpha + \beta + \gamma = \kappa + \lambda + \omega \tag{A1}$$

are non-trivial. Now consider a non-trivial example: let $\alpha = \kappa = \frac{1}{2}, \beta = \lambda = -\frac{3}{2}, \gamma = \omega = -\frac{3}{2}$, and represent $\mathbf{S}(\mathbf{S}^{-1})$ with a left (right) crossing; then the term $S_i S_{i+1}^{-1} S_i$ from $\Theta_3$ is represented diagrammatically as

$S_i S_{i+1}^{-1} S_i =$  (A2)

Consider the first crossing, . Since the total pseudospin is $-1$, we look for non-zero elements in $(\mathbf{A}_{-1})^{(*,*)}_{(1/2,-3/2)}$; there are only three of them, i.e., $(*,*) = (-\frac{3}{2}, \frac{1}{2}), (-\frac{7}{2}, \frac{5}{2})$, or $(-\frac{11}{2}, \frac{9}{2})$, and the corresponding matrix elements are $t^2$, $tw$ and $-t^2w$, respectively. (See equations (3.9) and (2.3).) Thus the corresponding diagrammatic expansion is




(A3)

Next, we consider the second and third crossings in a similar manner; then the other terms in (6.6) are also expanded in the same way. All the expansions are substituted in (6.5) for cases (a), (b) and (c) to see if they are satisfied. The invalid cases are excluded once a contradiction is found, but the valid case must be satisfied by all possible in- and out-states. The procedure is straightforward but tedious.

In fact, the present model has a typical quantum group structure for case (a), based on Jimbo's loop algebra consideration with the largest root [6]. The proof that (6.10) satisfies the YBE, based on general QG consideration, was given in [19]. Thus the diagrammatic expansion shown here can be considered a direct check of the general theory of Yang–Baxterisation.

## References


[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[2] Faddeev L D *Integrable Models in (1+1)-dimensional Quantum Field Theory (Les Houches session 39)* p 536
[3] Kulish P P and Sklyanin E K 1982 *Springer Lecture Notes in Physics* vol 151 (Berlin: Springer) p 61
[4] De Vega H J 1989 *Int. J. Mod. Phys.* A 4 2371
[5] Jimbo M 1989 *Int. J. Mod. Phys.* A 4 3759
[6] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
Wadati M, Deguchi T and Akutsu Y 1989 *Phys. Rep.* **180** 247
[7] Jimbo M 1986 *Commun. Math. Phys.* **102** 537
[8] Kuniba A 1989 *Preprint* Kyoto University
[9] Reshetikhin N Yu *LOMI Report* E-4-87, E-14-87
[10] Witten E 1989 *Commun. Math. Phys.* **121** 351
[11] Ge M L, Li Y Q and Xue K 1990 *Int. J. Mod. Phys.* **23** 619
[12] Ge M L, Wu Y S and Xue K 1989 *Preprint*, SUNY, Stony Brook, NY
[13] Ge M L, Li Y Q, Wang L Y and Xue K 1990 *J. Phys. A: Math. Gen.* **23** 605
[14] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[15] Akutsu Y and Wadati M 1987 *J. Phys. Soc. Japan* **56** 3039
Turaev V G 1988 *Invent. Math.* **92** 527
[16] Murakami J 1987 *Osaka J. Math.* **24** 745
Ge M L, Wang L Y, Xue K and Wu Y S 1989 *Int. J. Mod. Phys.* **4** 3351
[17] The idea of Baxterisation was first presented by V F R Jones; see 1989 *Proc. Lake Tahoe Conf.*
[18] Couture M, Ge M L and Lee H C 1989 *Preprint* AECL, Chalk River, Canada
[19] Faddeev L, Takhtajan L and Reshetikhin Yu 1989 *Alg. Anal.* **1** 178 (in Russian)
[20] Ge M L, Lee H C and Wang L Y 1989 *Preprint* AECL, Chalk River, Canada
[21] Karowski M 1979 *Nucl. Phys.* B **153** 244
[22] Kullisch P P, Reshetikhin N Yu and Sklyanin E K 1981 *Lett. Math. Phys.* **5** 393
[23] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[24] Couture M, Cheng Y, Ge M L and Xue K *Preprint*, SUNY, Stony Brook, NY ITPSB-90-005